AUSTRALIAN MATHS TRUST

Australian Mathematical Olympiad 2019

DAY 1
Tuesday, 5 February 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all real numbers r for which there exists exactly one real number a such that when

$$(x+a)(x^2+rx+1)$$

is expanded to yield a cubic polynomial, all of its coefficients are greater than or equal to zero.

2. For each positive integer n, the nth *triangular number* is the sum of the first n positive integers. Let a, b, c be three consecutive triangular numbers with $a < b < c$.

Prove that if $a+b+c$ is a triangular number, then b is three times a triangular number.

3. Let A, B, C, D, E be five points in order on a circle $\mathcal{K}$. Suppose that $AB = CD$ and $BC = DE$. Let the chords AD and BE intersect at the point P.

Prove that the circumcentre of triangle AEP lies on $\mathcal{K}$.

4. Let Q be a point inside the convex polygon $P_1P_2\cdots P_{1000}$. For each $i = 1, 2, \ldots, 1000$, extend the line P_iQ until it meets the polygon again at a point X_i. Suppose that none of the points $X_1, X_2, \ldots, X_{1000}$ is a vertex of the polygon.

Prove that there is at least one side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

Australian Mathematical Olympiad 2019

DAY 2
Wednesday, 6 February 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. A *fancy triangle* is an equilateral triangular array of integers such that the sum of the three numbers in any unit equilateral triangle is a multiple of 3. For example,

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 2 & \\ 5 & & 7 & & 3 \end{array}$$

is a fancy triangle with three rows because the sum of the numbers in each of the following four unit equilateral triangles is a multiple of 3.

$$\begin{array}{ccc} & 1 & \\ 0 & & 2 \end{array} \qquad \begin{array}{ccc} & 0 & \\ 5 & & 7 \end{array} \qquad \begin{array}{ccc} 0 & & 2 \\ & 7 & \end{array} \qquad \begin{array}{ccc} & 2 & \\ 7 & & 3 \end{array}$$

Suppose that a fancy triangle has ten rows and that exactly n of the numbers in the triangle are multiples of 3.

Determine all possible values for n.

6. Let $\mathcal{K}$ be the circle passing through all four corners of a square $ABCD$. Let P be a point on the minor arc CD, different from C and D. The line AP meets the line BD at X and the line CP meets the line BD at Y. Let M be the midpoint of XY.

Prove that MP is tangent to $\mathcal{K}$.

7. Akshay writes a sequence $a_1, a_2, \ldots, a_{100}$ of integers in which the first and last terms are equal to 0. Except for the first and last terms, each term a_i is larger than the average of its neighbours a_{i-1} and a_{i+1}.

What is the smallest possible value for the term a_{19}?

8. Let $n = 16^{3^r} - 4^{3^r} + 1$ for some positive integer r.

Prove that $2^{n-1} - 1$ is divisible by n.

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2019 Australian Mathematical Olympiad Solutions

1. Find all real numbers r for which there exists exactly one real number a such that when

$$(x+a)(x^2+rx+1)$$

is expanded to yield a cubic polynomial, all of its coefficients are greater than or equal to zero.

Solution 1 (Angelo Di Pasquale)

Answer: $r = -1$.

Expanding the brackets, we see that we want the following three inequalities to be true.

$$a \geq 0 \quad \text{(constant term)} \tag{1}$$
$$ar + 1 \geq 0 \quad \text{(coefficient of } x\text{)} \tag{2}$$
$$a + r \geq 0 \quad \text{(coefficient of } x^2\text{)} \tag{3}$$

If $r \geq 0$, then any $a \geq 0$ satisfies (1), (2), and (3).

It remains to address $r < 0$. In this case note that (3) immediately implies (1). So we only need to consider (2) and (3). Since $r < 0$, inequalities (2) and (3) are equivalent to the following.

$$a \leq -\frac{1}{r} \tag{4}$$
$$a \geq -r \tag{5}$$

Hence, we seek all values of $r < 0$ such that there is exactly one real number a satisfying

$$-r \leq a \leq -\frac{1}{r}. \tag{6}$$

Thus $-r = -\frac{1}{r}$, which implies $r = \pm 1$. Since $r < 0$ we have $r = -1$. This implies that the only corresponding value of a is $a = 1$.

It only remains to observe that

$$(x+1)(x^2-x+1) = x^3+1,$$

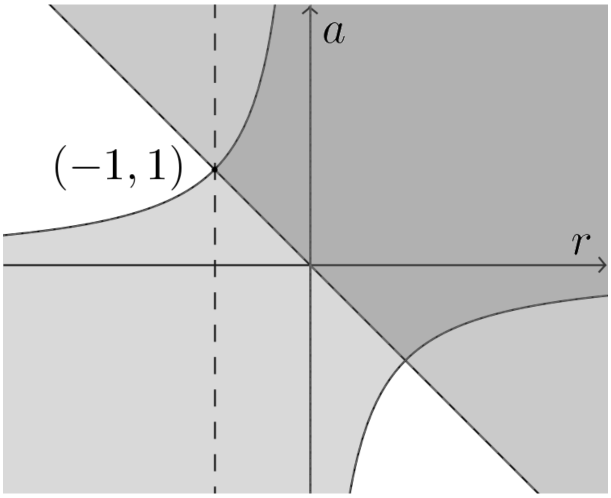
which has no negative coefficients.

Solution 2 (Alan Offer)

Expanded, the cubic is

$$x^3 + (a+r)x^2 + (ar+1)x + a.$$

Plotted on a Cartesian plane with a on the horizontal axis and r on the vertical axis, the condition that $a + r \geq 0$ is satisfied by the points in the region above and right of the line $a + r = 0$. Similarly, the condition that $ar + 1 \geq 0$ corresponds to the region between the two branches of the hyperbola $r = -1/a$.



The intersection of these two regions is then where both of the coefficients $a+r$ and $ar+1$ are non-negative, so we are being asked for the horizontal coordinates r at which a vertical line meets this region in exactly one point, and this occurs at $r = -1$, where the line and the hyperbola meet at $(-1, 1)$. (Notice that the coefficient a is then also non-negative.)

2. For each positive integer n, the nth *triangular number* is the sum of the first n positive integers. Let a, b, c be three consecutive triangular numbers with $a < b < c$.

   Prove that if $a + b + c$ is a triangular number, then b is three times a triangular number.

Solution 1 (Mike Clapper)

Let $T_m = T_{n-1} + T_n + T_{n+1}$.

Then $\frac{m}{2}(m+1) = \frac{n}{2}(n-1) + \frac{n}{2}(n+1) + \frac{n+1}{2}(n+2)$ which simplifies to $3(n^2+n)+2 = m^2 + m$.
Considering this equation modulo 3, we see that the LHS $\equiv 2 \pmod 3$.
This is only possible if $m \equiv 1 \pmod 3$ so we can let $m = 3s + 1$ for some integer s.

Hence, $3(n^2 + n) + 2 = (3s + 1)(3s + 2)$ giving $n^2 + n = 3(s^2 + s)$ and $T_n = 3T_s$.

Solution 2 (Ivan Guo)

Instead of triangular numbers, it suffices to double everything and work only with numbers of the form $n(n + 1)$ where $n \geq 1$. The required condition can be rewritten as

$$n(n-1) + n(n+1) + (n+1)(n+2) = (m+1)(m+2) \iff 3n^2 + 3n = m^2 + 3m.$$

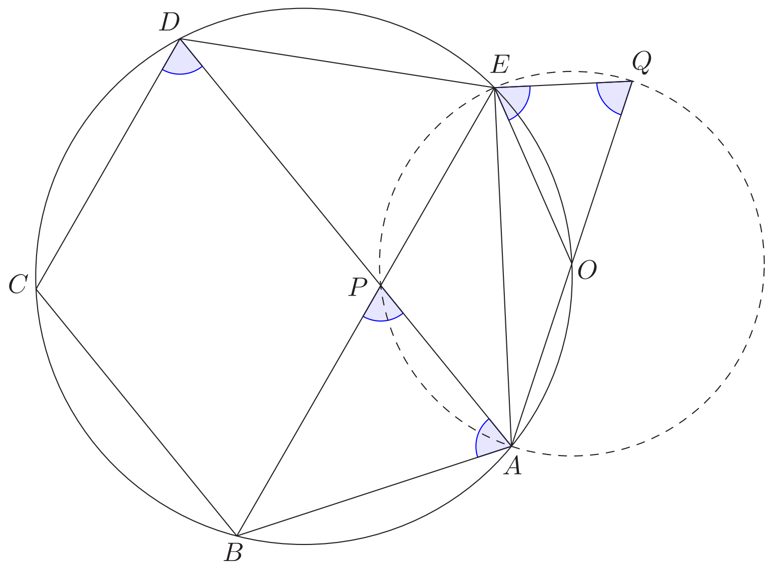
So $3 \mid m$. Writing $m = 3s$ yields $n^2 + n = 3(s^2 + s)$, as required. (Note that we need to check $s \geq 1$ but this is clear since both sides are positive here.)

3. Let A, B, C, D, E be five points in order on a circle $\mathcal{K}$. Suppose that $AB = CD$ and $BC = DE$. Let the chords AD and BE intersect at the point P.

   Prove that the circumcentre of triangle AEP lies on $\mathcal{K}$.

Solution 1 (Angelo Di Pasquale)

Since $AB = CD$ and $ABCD$ is cyclic, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Similarly, $BE \parallel CD$.



Let O be the midpoint of arc AE of circle $ABCDE$. Thus $OA = OE$. Let Q be second intersection point of line AO with circle AEP. Let $x = \angle BPA$. We calculate the following angles.

$\angle CDA = x$	$(BE \parallel CD)$
$\angle DAB = x$	(isosceles trapezium $ABCD$)
$\angle EQA = x$	($AQEP$ cyclic)
$\angle ABP = 180^\circ - 2x$	(angle sum $\triangle ABP$)
$\angle QOE = 180^\circ - 2x$	($ABEO$ cyclic)
$\angle OEQ = x$	(angle sum $\triangle OQE$)

Hence, $\triangle OQE$ is isosceles with $OQ = OE$. Since $OQ = OE = OA$, it follows circle AEQ has centre O. Since P also lies on this circle, we may conclude that O is the circumcentre of $\triangle AEP$.

Solution 2 (Alice Devillers)

We need to prove that the centre O of the circumcircle to AEP satisfies $\angle AOE = 180 - \angle ADE$.

We will repeatedly use the angles intercepting arcs of the same length are the same: for instance $\angle ABC = \angle BCD = \angle CDE$.

Since the sum of the angles in a pentagon is $540°$, so

$$\begin{aligned}540° &= \angle ABC + \angle BCD + \angle CDE + \angle DEA + \angle EAB \\ &= 3\angle CDE + \angle DEB + \angle BEA + \angle EAD + \angle DAB \\ &= 3\angle CDE + 180° - \angle DCB + \angle BEA + \angle EAD + 180° - \angle DCB \\ &= 360° + \angle CDE + 180° - \angle APE.\end{aligned}$$

Thus, $\angle CDE = \angle APE$.

Because of the circumcircle, $\angle AOE = 360° - 2\angle APE = 360° - 2\angle CDE$. On the other hand,

$$\angle ADE = \angle CDE - \angle ADC = \angle CDE - (180° - \angle ABC) = 2\angle CDE - 180°.$$

Hence, $\angle AOE + \angle ADE = 180°$ and we are done.

Solution 3 (Angelo Di Pasquale)

With notation in solution 1, we have $\angle APE = 180° - x$ and $\angle EOA = 2x$. Thus $\angle AEO$ (reflex) $= 360° - 2x = 2\angle APE$. Consider any point X that satisfies the following.

- X and P lie on opposite sides of line AE.
- X lies on the perpendicular bisector of AE.
- $\angle AXE$(reflex) $= 2\angle APE$.

There is only one point X that has these properties. This is because as X moves on the perpendicular bisector of AE away from (closer to) AE, the reflex angle AXE gets larger (smaller). The circumcentre of $\triangle AEP$ and point O both have the above properties. Hence, O is the circumcentre of $\triangle AEP$.

Solution 4 (Angelo Di Pasquale)

(Variation on the alternative solution) Let $x = \angle BPA$. Then $\angle CDA = x$ since $CD \parallel BE$ from isosceles trapezium $BCDE$. Also $\angle DAB = x$ from isosceles trapezium $ABCD$. From the angle sum in $\triangle ABP$, we deduce $\angle ABE = 180° - 2x$.

We also have $\angle APE = 180° - x$. Let O be the circumcentre of $\triangle AEP$. Thus reflex angle $\angle AOE = 2\angle APE = 360° - 2x$, and so $\angle EOA = 2x$. Since $\angle ABE + \angle EOA = 180°$, it follows that $ABEO$ is cyclic. Thus O lies on circle(ABE) = circle($ABCDE$).

Solution 5 (Ivan Guo)

The given length conditions imply that $ABCD$ and $BCDE$ are isosceles trapezia, while $BCDP$ is a parallelogram. Hence, let $\angle ABC = \angle BCD = \angle CDE = \angle EPA = \theta$. Construct O to be the midpoint of the arc AE. Since in a cyclic hexagon, the three non-adjacent angles add up to $360°$, we have $360° - \angle EOA = 2\theta = 2\angle EPA$. Therefore, O is the circumcentre of EPA.

Solution 6 (Daniel Mathews)

Let the given circle be Γ, with centre O, and let $a = \angle DAEandb = \angle BEA$. Then a and b are the angles subtended by the arcs DE and AB respectively; note $a, b < 90°$. As $AB = CD$ then $\angle CED = b$, and as $BC = DE$ then $\angle BEC = a$.

Now AE subtends $\angle APE = 180 - \angle AEP - \angle EAP = 180 - a - b$ at P, which is obtuse. Hence, AE subtends $a + b$ at points of Γ on the other side of AE from P, and subtends $2a + 2b$ at O. Thus O lies on the other side of AE from P, and satisfies $\angle AOE = 2a + 2b$.

On the other hand, AE subtends an angle of $180° - 2a - 2b$ at D, since

$$\angle ADE = 180 - \angle DAE - \angle AED = 180 - \angle DAE - \angle AEP - \angle BEC - \angle CED = 180 - 2a - 2b,$$

and hence, subtends $2a + 2b$ at points of Γ on the other side of AE from P. Thus O lies on Γ.

Solution 7 (Kevin McAvaney)

From isosceles trapezia $ABCD$ and $BCDE$, triangles ABP and EDP are isosceles and equiangular. Let DO be the perpendicular bisector of EP with O on the circumcircle of $ABCDE$. Then DO bisects angle PDE. Angles ODE and OBE are equal. Hence, BO bisects angle ABP. Therefore, BO is the perpendicular bisector of AP. Hence, O the circumcentre of triangle AEP.

Solution 8 (Alan Offer)

Let O be the centre of the circumcircle of triangle AEP. In terms of directed angles, it follows that $\angle AOE = 2\angle APE$. Now O is on the circumcircle of $ABCDE$ if $\angle AOE = \angle ABE$, so it suffices to show that $\angle ABE = 2\angle APE$. With this in mind, we have

$$\begin{aligned}
2\angle APE &= 2\angle ABE + 2\angle DAB && \text{(exterior angle of } \triangle ABP\text{)} \\
&= \angle ABE + \angle ACE + 2\angle DAB && (ABCE \text{ cyclic}) \\
&= \angle ABE + \angle ACE + (\angle DAC + \angle CAB) + \angle DAB \\
&= \angle ABE + \angle DAB + (\angle ACE + \angle BCA + \angle ECD) && (\text{arcs } DC = BA \text{ and } CB = ED) \\
&= \angle ABE + (\angle DAB + \angle BCD) \\
&= \angle ABE && (ABCD \text{ cyclic}).
\end{aligned}$$

4. Let Q be a point inside the convex polygon $P_1P_2 \cdots P_{1000}$. For each $i = 1, 2, \ldots, 1000$, extend the line P_iQ until it meets the polygon again at a point X_i. Suppose that none of the points $X_1, X_2, \ldots, X_{1000}$ is a vertex of the polygon.

   Prove that there is at least one side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

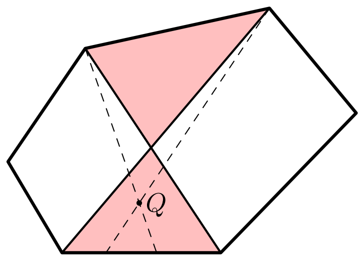
Solution 1

Since Q does not lie on a diagonal of the polygon, each of the points $X_1, X_2, \ldots, X_{1000}$ lies on the interior of a side of the polygon. Suppose that X_1 lies on the side P_iP_{i+1}. Without loss of generality, we may assume that $i \leq 500$; otherwise, we could relabel the vertices in the opposite orientation instead.

Then the points $P_2, P_3, \ldots, P_i$ lie on one side of the line P_1Q, which means that the points $X_2, X_3, \ldots, X_i$ must lie on the other side of the line P_1Q. So the i points $X_1, X_2, X_3, \ldots, X_i$ must lie on the $1001 - i$ sides $P_iP_{i+1}, P_{i+1}P_{i+2}, \ldots, P_{1000}P_1$. Furthermore, no other point X_j can lie on one of these sides, since they lie on the other side of the line P_1Q. However, since $i \leq 500$, we have $i < 1001 - i$. It follows that there must be at least one of the sides $P_iP_{i+1}, P_{i+1}P_{i+2}, \ldots, P_{1000}P_1$ that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

Solution 2 (Angelo Di Pasquale)

Define a *butterfly* to be the region formed by the two triangles cut out by a pair of consecutive main diagonals of the polygon. If Q lies inside a butterfly, then it is easy to see that the conclusion of the problem is true since a line that enters a triangle must exit it somewhere.



To finish, it suffices to prove that the point Q lies inside a butterfly. For any directed line AB, we define its *positive* side to be the half-plane of points X such that $0 < \angle BAX < 180°$. We also define its *negative* side to be the half-plane of points X such that $180° < \angle BAX < 360°$. In both cases, the angle is directed anticlockwise modulo $360°$.

Without loss of generality, suppose that Q lies on the positive side of the directed line P_0P_{500}, where we consider all subscripts modulo 1000. Then Q lies on the negative side of the directed line $P_{500}P_0$. Hence, there exists an integer i with $0 \leq i \leq 499$ such that Q lies on the positive side of P_iP_{i+500} but on the negative side of $P_{i+1}P_{i+501}$. Thus, Q lies inside the butterfly defined by P_iP_{i+500} and $P_{i+1}P_{i+501}$.

Solution 3 (Kevin McAvaney)

We will prove the statement more generally for a convex $2m$-gon. Suppose that each side of the polygon contains at least one of the points $X_1, X_2, \ldots, X_{2m}$ on its interior. Then each side contains exactly one of the points $X_1, X_2, \ldots, X_{2m}$ on its interior. Otherwise, one of the lines through Q passes through an interior point of at least two polygon sides and this contradicts the convexity of the polygon.

Label the lines through Q in clockwise order $L_1, L_2, L_3, \ldots, L_{2m}$. Label the vertices of the polygon in clockwise order $P_1, P_2, P_3, \ldots, P_{2m}$. Without loss of generality, suppose that L_1 passes through P_1. Then L_2 passes through an interior point of the side P_1P_2 and L_3 passes through P_2. Hence, L_4 passes through an interior point of side P_2P_3 and L_5 passes through P_3. Continuing in the same manner, we see that only the lines indexed by odd integers pass through vertices of the polygon and this produces the desired contradiction.

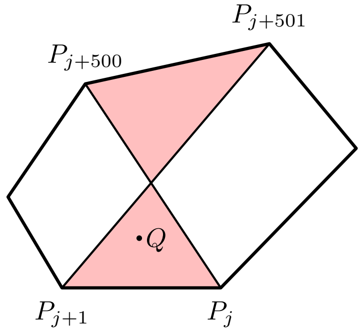
Solution 4 (Chaitanya Rao)

For notational convenience let $P_{1000+i} = P_i$ for $i \in \{1, 2, \ldots, 1000\}$. The diagonal P_iP_{i+500} joining opposite vertices divides the convex polygon into the following two 501-gons: $P_1P_2 \cdots P_iP_{i+500}P_{i+501} \cdots P_{1000}$ and $P_iP_{i+1}P_{i+2} \cdots P_{i+500}$. Since Q is not on a diagonal and the original polygon is convex, Q lies inside one of these polygons and outside the other.

Define a function $f \colon \{1, 2, \ldots, 1000\} \to \{0, 1\}$ by

$$f(i) = \begin{cases} 1, & \text{if } Q \text{ lies inside } P_1P_2 \cdots P_iP_{i+500}P_{i+501} \cdots P_{1000}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f(i) = 1 - f(i + 500)$, since Q is inside exactly one of the two 501-gons $P_1P_2 \cdots P_iP_{i+500}P_{i+501} \cdots P_{1000}$ and $P_iP_{i+1} \cdots P_{i+500}$. Hence, the function f is not constant and there exists some j for which $f(j) = 0$ and $f(j+1) = 1$, as shown in the following diagram.



We then find that segment P_jP_{j+1} contains both X_{j+500} and X_{j+501} since they are the bases of internal cevians of triangles $P_{j+500}P_jP_{j+1}$ and $P_{j+501}P_jP_{j+1}$, respectively. Note that both triangles have Q in the interior of their intersection. Since Q is not on a diagonal, none of the points X_i is a vertex of the the polygon and we conclude that there exists another side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

Solution 5 (Ian Wanless)

Consider the diagonal $d = P_1P_{m+1}$. By assumption, Q does not lie on d. Assume that Q lies on the same side of d as P_{2m} does, since the other case is equivalent after relabelling the vertices. Let $1 \leq i \leq m + 1$ and note that the ray from P_i to Q hits d before it hits Q. This means that the point X_i is on the same side of d as Q. But this means that the

$m+1$ points $X_1, X_2, \ldots, X_{m+1}$ lie on the m sides $P_{m+1}P_{m+2}, \ldots, P_{2m-1}P_{2m}, P_{2m}P_1$. By the pigeonhole principle, at least two of points $X_1, X_2, \ldots, X_{m+1}$ lie on the same side of the polygon. By a second application of the pigeonhole principle, it follows that there is some side of the polygon that contains none of the points $X_1, X_2, \ldots, X_{2m}$.

5. A *fancy triangle* is an equilateral triangular array of integers such that the sum of the three numbers in any unit equilateral triangle is a multiple of 3. For example,

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 2 & \\ 5 & & 7 & & 3 \end{array}$$

is a fancy triangle with three rows because the sum of the numbers in each of the following four unit equilateral triangles is a multiple of 3.

$$\begin{array}{ccc} & 1 & \\ 0 & & 2 \end{array} \qquad \begin{array}{ccc} & 0 & \\ 5 & & 7 \end{array} \qquad \begin{array}{ccc} 0 & & 2 \\ & 7 & \end{array} \qquad \begin{array}{ccc} & 2 & \\ 7 & & 3 \end{array}$$

Suppose that a fancy triangle has ten rows and that exactly n of the numbers in the triangle are multiples of 3.

Determine all possible values for n.

Solution (Angelo Di Pasquale)

Answers: $n = 0$, 18, 19, or 55

Consider the four numbers in any two unit equilateral triangles that share a common edge as shown in the diagram.

$$\begin{array}{ccc} & u & \\ v & & w \\ & x & \end{array}$$

Since $u + v + w \equiv 0 \equiv v + w + x \pmod 3$, it follows that $u \equiv x \pmod 3$. Using this observation we deduce that if we reduce the entries of the triangle modulo 3, it takes the following form.

$$\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & u & & & & & & & & & \\
 & & & & & & & & v & & w & & & & & & & & \\
 & & & & & & & w & & u & & v & & & & & & & \\
 & & & & & & u & & v & & w & & u & & & & & & \\
 & & & & & v & & w & & u & & v & & w & & & & & \\
 & & & & w & & u & & v & & w & & u & & v & & & & \\
 & & & u & & v & & w & & u & & v & & w & & u & & & \\
 & & v & & w & & u & & v & & w & & u & & v & & w & & \\
 & w & & u & & v & & w & & u & & v & & w & & u & & v & \\
u & & v & & w & & u & & v & & w & & u & & v & & w & & u
\end{array}$$

Note that the triangular array is fancy if and only if $3 \mid u + v + w$. Reducing modulo 3, we have the cases $(u, v, w) = (0, 0, 0), (1, 1, 1), (2, 2, 2)$, or any permutation of $(0, 1, 2)$.

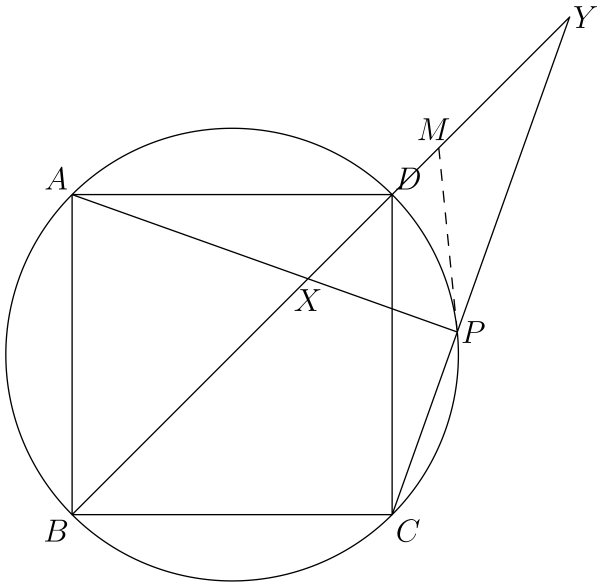
- If $(u, v, w) = (0, 0, 0)$, then $n = 55$.
- If $(u, v, w) = (1, 1, 1)$ or $(2, 2, 2)$, then $n = 0$.
- If $(u, v, w) = (0, 1, 2)$ or $(0, 2, 1)$, then $n = 19$.
- If $(u, v, w) = (1, 0, 2)$ or $(1, 2, 0)$ or $(2, 1, 0)$ or $(2, 0, 1)$, then $n = 18$.

6. Let $\mathcal{K}$ be the circle passing through all four corners of a square $ABCD$. Let P be a point on the minor arc CD, different from C and D. The line AP meets the line BD at X and the line CP meets the line BD at Y. Let M be the midpoint of XY.

   Prove that MP is tangent to $\mathcal{K}$.

Solution 1

By the converse of the alternate segment theorem, it suffices to prove that $\angle MPA = \angle ABP$.



Let $\angle AXB = \angle MXP = \theta$. Since AC is a diameter of the circle, $\angle APC = \angle APY = 90°$. So M is the midpoint of the hypotenuse of the right-angled triangle XYP. It follows that $MX = MP$, so $\angle MPA = \angle MPX = \angle MXP = \theta$.

Now observe that $\angle AXD = 180° - \theta$ and $\angle XDA = 45°$, so $\angle DAP = \angle DAX = \theta - 45°$. By cyclic quadrilateral $ABPD$, we have $\angle DBP = \angle DAP = \theta - 45°$. Therefore, $\angle ABP = \angle ABD + \angle DBP = 45° + (\theta - 45°) = \theta$.

So we have shown that $\angle MPA = \angle ABP = \theta$, as required.

Solution 2 (Alice Devillers)

Pick coordinates such that $A = (0, -1)$, $B = (-1, 0)$, $C = (0, 1)$, $D = (0, 1)$, so $P = (\cos\theta, \sin\theta)$ where θ is between 0 and $\pi/2$. We easily compute the equations of AP: $y+1 = \frac{\sin\theta+1}{\cos\theta}x$ and CP: $y-1 = \frac{\sin\theta-1}{\cos\theta}x$, while BD is just $y = 0$. Thus $X = (\frac{\cos\theta}{\sin\theta+1}, 0)$ and $Y = (-\frac{\cos\theta}{\sin\theta-1}, 0)$. The middle point M is $X = (\frac{1}{\cos\theta}, 0)$ (here we used $\sin^2\theta - 1 = -\cos^2\theta$ and $\cos\theta \neq 0$). If we take the dot product of the vectors OP and MP, we get 0 so MP is tangent to the circle of radius 1 centred at O.

Solution 2 (Angelo Di Pasquale)

Let $O = AC \cap BD$. Note that $AC \perp BD$, and so $\angle AOY = 90°$. Also $\angle AP \perp PC$ because AC is a diameter of $\mathcal{K}$. Thus $\angle APY = 90° = \angle AOY$, and so $AOPY$ is cyclic.

As $\angle XPY = 90^\circ$ and M is the midpoint of XY, we have M is the centre of circle PXY. Thus $MX = MP = MY$.

From $MY = MP$ and cyclic $AOPY$, we find $\angle MPY = \angle PYO = \angle PXO = \angle PXC$, and so by the alternate segment theorem, MP is tangent to $\mathcal{K}$ at P.

Solution 3 (Ivan Guo)

Let Q be the reflection of P about AC, so $PQ \| BD$. Then since $APCQ$ is a cyclic kite, the points A, P, C, Q are harmonic. Projecting them from P onto BD yields X, M', Y, ∞ where M' is the intersection of the tangent at P with BD. Since X, M', Y, ∞ are harmonic, then M' must be the midpoint of XY.

Solution 4 (Ivan Guo)

Since $ABCD$ is a square, A, B, C, D are harmonic. Projecting from P onto BD yields the harmonic points B, X, D, Y. Via a standard length calculation on the line BD, we immediately get $MX^2 = MD \times MB$. Since $AP \perp PY$, $MX = MP$ and the required tangency follows by power of a point.

Solution 5 (Ivan Guo)

Since $ABCD$ is a square, AP and CP are internal and external angle bisectors of $\angle BPD$. By the angle bisector theorem, we see that the circles DPB and XPY are circles of Apollonius. It is well-known that circles of Apollonius are orthogonal, hence the required tangency.

Solution 6 (Ivan Guo)

Let AY meet the circle at R. Apply the central projection that sends the line through Y perpendicular to BD to infinity while maintaining the circle. Then $A'R'P'C'$ is rectangle, hence X' is the new centre of the circle. Furthermore $\infty' C'$ is a tangent to the circle. But since harmonic points are preserved under central projections, X' is the midpoint of $M'\infty'$. By symmetry and $B'D' \| P'C'$, we must have $M'P'$ being a tangent to the circle.

Solution 7 (Kevin McAvaney)

Let O be the centre of the circle. We show that OP and MP are perpendicular.

Angle CPA = angle CDA = 90 degrees. So XPY is a right-angled triangle and M is therefore the centre of its circumcircle. Hence MP = MY. Since ABCD is a square, O is the intersection of its diagonals and they are perpendicular.

So we have angle MPY = angle MYP = angle OAP = angle OPA. Hence angle OPM = angle OPA + angle APM = angle MPY + angle APM = 90 degrees, as required.

Solution 8 (Alan Offer)

This problem can be handled fine with coordinates. Choose coordinates so that $A = (-1, 0)$, $B = (0, -1)$, $C = (1, 0)$, and $D = (0, 1)$. Then $P = (u, v)$ with $u^2 + v^2 = 1$. Also,

$X = (0, s)$ and $Y = (0, t)$ for some numbers s and t. As the slope of AX is equal to the slope of AP, we obtain $s = v/(1 + u)$. As the slope of CY is equal to the slope of CP, we have $t = v/(1-u)$. The midpoint of XY is then at $M = (0, \frac{1}{2}(s+t)) = (0, v/(1-u^2)) = (0, 1/v)$.

Calling the origin O, the product of the slopes of MP and OP is

$$\frac{v - 1/v}{u} \times \frac{v}{u} = \frac{v^2 - 1}{u^2} = \frac{-u^2}{u^2} = 1.$$

Hence MP is perpendicular to the radius OP and so is tangent to the circle.

Solution 9 (Chaitanya Rao)

As in the official solution we have $\angle MPA = \angle DXP$. If O is the centre of $\mathcal{K}$, then by the angle between intersecting chords theorem, $\angle DXP = \frac{1}{2}(\angle AOB + \angle DOP) = 45^\circ + \frac{1}{2}\angle DOP = \angle ABD + \angle DBP = \angle ABP$. Hence $\angle MPA = \angle ABP$ and by the converse of the alternate segment theorem MP is tangent to $\mathcal{K}$.

7. Akshay writes a sequence $a_1, a_2, \ldots, a_{100}$ of integers in which the first and last terms are equal to 0. Except for the first and last terms, each term a_i is larger than the average of its neighbours a_{i-1} and a_{i+1}.

   What is the smallest possible value for the term a_{19}?

Solution

Let $d_i = a_{i+1} - a_i$ for $i = 1, 2, 3, \ldots, 99$, so that $a_j = d_1 + d_2 + \cdots + d_{j-1}$ for $j = 2, 3, \ldots, 100$. The conditions of the problem are equivalent to the fact that $d_1 > d_2 > \cdots > d_{99}$ are integers and

$$d_1 + d_2 + \cdots + d_{99} = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{100} - a_{99}) = a_{100} - a_1 = 0.$$

Observe that we can take $d_i = 50 - i$ for $i = 1, 2, 3, \ldots, 99$, which yields

$$\begin{aligned} a_{19} &= (a_{19} - a_{18}) + (a_{18} - a_{17}) + \cdots + (a_2 - a_1) \\ &= d_{18} + d_{17} + \cdots + d_1 \\ &= (50 - 18) + (50 - 17) + \cdots + (50 - 1) \\ &= 18 \times 50 - \frac{18 \times 19}{2} \\ &= 729. \end{aligned}$$

We will now show that this is the smallest possible value for a_{19}. For the sake of contradiction, suppose that $a_{19} < 729$. Then

$$\begin{aligned} 729 > a_{19} &= d_{18} + d_{17} + \cdots + d_1 \\ &\geq (d_{18}) + (d_{18} + 1) + \cdots + (d_{18} + 17) = 18d_{18} + \frac{17 \times 18}{2}. \end{aligned}$$

This leads to $d_{18} < 32$.

However, we also have

$$\begin{aligned} -729 < -a_{19} &= -(d_1 + d_2 + \cdots + d_{18}) = d_{19} + d_{20} + \cdots + d_{99} \\ &\leq (d_{18} - 1) + (d_{18} - 2) + \cdots + (d_{18} - 81) = 81d_{18} - \frac{81 \times 82}{2}. \end{aligned}$$

This leads to $d_{18} > 32$, which contradicts the inequality obtained earlier. Therefore, we can conclude that the smallest possible value for the term a_{19} is 729.

8. Let $n = 16^{3^r} - 4^{3^r} + 1$ for some positive integer r.

   Prove that $2^{n-1} - 1$ is divisible by n.

Solution 1 (Angelo Di Pasquale)

Observe that n has the form $y^2 - y + 1$, where $y = 4^{3^r}$. Thus, $4^{3^{r+1}} + 1 = y^3 + 1 = n(y+1)$. Therefore,

$$\begin{aligned} & 4^{3^{r+1}} + 1 \equiv 0 \pmod{n} \\ \Rightarrow\quad & 2^{2\cdot 3^{r+1}} \equiv -1 \pmod{n} \\ \Rightarrow\quad & 2^{4\cdot 3^{r+1}} \equiv 1 \pmod{n}. \end{aligned} \tag{1}$$

To show that $2^{n-1} \equiv 1 \pmod{n}$, it suffices to show that $4 \cdot 3^{r+1} \mid n - 1$, since if $n - 1 = 4 \cdot 3^{r+1}m$, then raising both sides of (1) to the power of m yields the result.

Since $n - 1 = 4^{3^r}(4^{3^r} - 1)$, it suffices to show that $3^{r+1} \mid 4^{3^r} - 1$. This can be done either by induction or by repeatedly factoring using the difference of perfect cubes.

Variant 1. (By induction)

For $r = 1$, it is easily verified that $3^2 \mid 4^3 - 1$.

Assume that $3^{r+1} \mid 4^{3^r} - 1$. Then

$$4^{3^{r+1}} - 1 = (4^{3^r})^3 - 1 = (4^{3^r} - 1)(16^{3^r} + 4^{3^r} + 1).$$

The inductive assumption tells us that 3^{r+1} divides the first bracket. The second bracket is congruent to $1^{3^r} + 1^{3^r} + 1 \equiv 0$ modulo 3. Thus, 3^{r+2} divides $4^{3^{r+1}} - 1$ and this completes the induction.

Variant 2. (By repeatedly factoring using the difference of perfect cubes)

$$\begin{aligned} 4^{3^r} - 1 &= (4^{3^{r-1}} - 1)(16^{3^{r-1}} + 4^{3^{r-1}} + 1) \\ &= (4^{3^{r-2}} - 1)(16^{3^{r-2}} + 4^{3^{r-2}} + 1)(16^{3^{r-1}} + 4^{3^{r-1}} + 1) \\ &\ \ \vdots \\ &= (4 - 1)\prod_{i=0}^{r-1}(16^{3^i} + 4^{3^i} + 1) \end{aligned}$$

Each bracket in the above factorisation is divisible by 3. Since there are $r + 1$ brackets, it follows that 3^{r+1} divides $4^{3^r} - 1$.

Solution 2 (Ivan Guo)

In the last part of the official solution, in order to prove $3^{r+1} | 4^{3^r} - 1$, it suffices to note that $\phi(3^{r+1}) = 2 \times 3^r$, thus $4^{3^r} = 2^{\phi(3^{r+1})} \equiv 1 \bmod 3^{r+1}$ by Euler's theorem.