Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2014

Junior Section (First Round)

Tuesday, 3 June 2014 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Let x, y and z be real numbers satisfying $x > y > 0$ and $z \neq 0$. Which of the inequalities below is not always true?

   (A) $x + z > y + z$ (B) $x - z > y - z$ (C) $xz > yz$ (D) $\frac{1}{y} + z > \frac{1}{x} + z$
   (E) $xz^2 > yz^2$

2. If the radius of a circle is increased by 100%, the area is correspondingly increased by how many percent?

   (A) 50% (B) 100% (C) 200% (D) 300% (E) 400%

3. If $a = \sqrt{7}$, $b = \sqrt{90}$, find the value of $\sqrt{6.3}$.

   (A) $\frac{7b}{a\sqrt{10}}$ (B) $\frac{b-7a}{10}$ (C) $\frac{10a}{b}$ (D) $\frac{ab}{100}$ (E) None of the above

4. Find the value of $\frac{1}{1-\sqrt[4]{5}} + \frac{1}{1+\sqrt[4]{5}} + \frac{2}{1+\sqrt{5}}$.

   (A) -1 (B) 1 (C) $-\sqrt{5}$ (D) $\sqrt{5}$ (E) None of the above

5. Andrew, Catherine, Michael, Nick and Sally ordered different items for lunch. These are (in no particular order): cheese sandwich, chicken rice, duck rice, noodles and steak. Find out what Catherine had for lunch if we are given the following information:

   1. Nick sat between his friend Sally and the person who ordered steak.
   2. Michael does not like noodles.
   3. The person who ate noodles is Sally's cousin.
   4. Neither Catherine, Michael nor Nick likes rice.
   5. Andrew had duck rice.

   (A) Cheese sandwich (B) Chicken rice (C) Duck rice (D) Noodles (E) Steak

6. At 2:40 pm, the angle formed by the hour and minute hands of a clock is $x°$, where $0 < x < 180$. What is the value of x?

   (A) 60° (B) 80° (C) 100° (D) 120° (E) 160°

7. In the figure below, each distinct letter represents a unique digit such that the arithmetic sum holds. If the letter L represents 9, what is the digit represented by the letter T?

```
    T E R R I B L E
+       N U M B E R
-------------------
    T H I R T E E N.
```

   (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

8. A regular cube is to have 2 faces coloured red, 2 faces coloured blue and 2 faces coloured orange. We consider two colourings to be the same if one can be obtained by a rotation of the cube from another. How many different colourings are there?

(A) 4 (B) 5 (C) 6 (D) 8 (E) 9

9. In $\triangle ABC$, $AB = AC$, $\angle BAC = 120°$, D is the midpoint of BC, and E is a point on AB such that DE is perpendicular to AB. Find the ratio $AE : BD$.

(A) $1:2$ (B) $2:3$ (C) $1:\sqrt{3}$ (D) $1:2\sqrt{3}$ (E) $2:3\sqrt{3}$

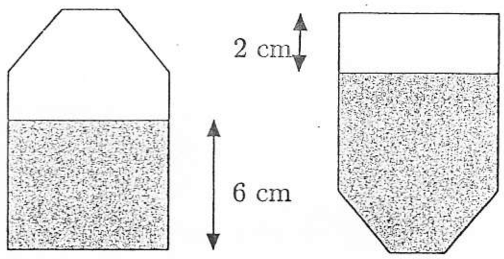
10. How many ways are there to add four positive odd numbers to get a sum of 22?

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

Short Questions

11. Successive discounts of 10% and 20% are equivalent to a single discount of x%. What is the value of x?

12. The diagram below shows the front view of a container with a rectangular base. The container is filled with water up to a height of 6 cm. If the container is turned upside down, the height of the empty space is 2 cm. Given that the total volume of the container is 28 cm^3, find the volume of the water in cm^3.



13. Let A be the solution of the equation

$$\frac{x-7}{x-8} - \frac{x-8}{x-9} = \frac{x-10}{x-11} - \frac{x-11}{x-12}.$$

Find the value of $6A$.

14. The sum of the two smallest positive divisors of an integer N is 6, while the sum of the two largest positive divisors of N is 1122. Find N.

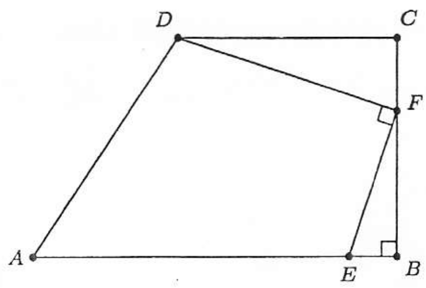
15. Let D be the absolute value of the difference of the two roots of the equation $3x^2 - 10x - 201 = 0$. Find $\lfloor D \rfloor$.

16. If m and n are positive real numbers satisfying the equation

$$m+4\sqrt{mn}-2\sqrt{m}-4\sqrt{n}+4n=3,$$

find the value of $\dfrac{\sqrt{m}+2\sqrt{n}+2014}{4-\sqrt{m}-2\sqrt{n}}$.

17. In the diagram below, $ABCD$ is a trapezium with $AB \parallel DC$ and $\angle ABC = 90°$. Points E and F lie on AB and BC respectively such that $\angle EFD = 90°$. If $CD + DF = BC = 4$, find the perimeter of $\triangle BFE$.



18. If p, q and r are prime numbers such that their product is 19 times their sum, find $p^2 + q^2 + r^2$.

19. John received a box containing some marbles. Upon inspecting the marbles, he immediately discarded 7 that were chipped. He then gave one-fifth of the marbles to his brother. After adding the remaining marbles to his original collection of 14, John discovered that he could divide his marbles into groups of 6 with exactly 2 left over or he could divide his marbles into groups of 5 with none left over. What is the smallest possible number of marbles that John received from the box?

20. Let N be a 4-digit number with the property that when all the digits of N are added to N itself, the total equals 2019. Find the sum of all the possible values of N.

21. There are exactly two ways to insert the numbers $1, 2$ and 3 into the circles

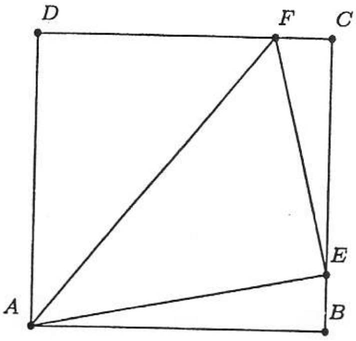
○ < ○ > ○

such that every order relation $<$ or $>$ between numbers in adjacent circles is satisfied. The two ways are ①< ③> ② and ②< ③> ①.

Find the total number of possible ways to insert the numbers $3, 14, 15, 9, 2$ and 6 into the circles below, such that every order relation $<$ or $>$ between the numbers in adjacent pairs of circles is satisfied.

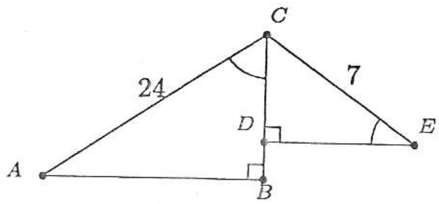
○ > ○ > ○ > ○ < ○ < ○.

22. Let $ABCD$ be a square of sides 8 cm. If E and F are variable points on BC and CD respectively such that $BE = CF$, find the smallest possible area of the triangle $\triangle AEF$ in cm^2.



23. If a, b and c are non-zero real numbers satisfying $a+2b+3c = 2014$ and $2a+3b+2c = 2014$, find the value of $\dfrac{a^2+b^2+c^2}{ac+bc-ab}$.

24. In the diagram below, $\triangle ABC$ and $\triangle CDE$ are two right-angled triangles with $AC = 24$, $CE = 7$ and $\angle ACB = \angle CED$. Find the length of the line segment AE.



25. The hypotenuse of a right-angled triangle is 10 and the radius of the inscribed circle is 1. Find the perimeter of the triangle.

26. Let x be a real number satisfying $\left(x+\dfrac{1}{x}\right)^2 = 3$. Evaluate $x^3 + \dfrac{1}{x^3}$.

27. For $2 \le x \le 8$, we define $f(x) = |x-2| + |x-4| - |2x-6|$. Find the sum of the largest and smallest values of $f(x)$.

28. If both n and $\sqrt{n^2 + 204n}$ are positive integers, find the maximum value of n.

29. Let $N = \overline{abcd}$ be a 4-digit perfect square that satisfies $\overline{ab} = 3 \cdot \overline{cd} + 1$. Find the sum of all possible values of N.

(The notation $n = \overline{ab}$ means that n is a 2-digit number and its value is given by $n = 10a+b$.)

30. Find the following sum:

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{29}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{29}\right)$$
$$+\left(\frac{3}{4}+\frac{3}{5}++\cdots+\frac{3}{29}\right)+\cdots+\left(\frac{27}{28}+\frac{27}{29}\right)+\frac{28}{29}.$$

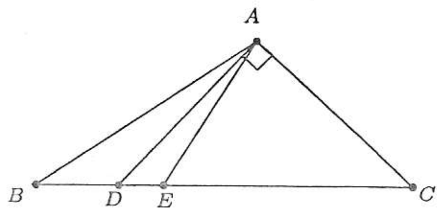
31. If $ax+by=7$, $ax^2+by^2=49$, $ax^3+by^3=133$, and $ax^4+by^4=406$, find the value of $2014(x+y-xy)-100(a+b)$.

32. For $a \geq \frac{1}{8}$, we define

$$g(a)=\sqrt[3]{a+\frac{a+1}{3}\sqrt{\frac{8a-1}{3}}}+\sqrt[3]{a-\frac{a+1}{3}\sqrt{\frac{8a-1}{3}}}.$$

Find the maximum value of $g(a)$.

33. In the diagram below, AD is perpendicular to AC and $\angle BAD = \angle DAE = 12°$. If $AB + AE = BC$, find $\angle ABC$.



34. Define S to be the set consisting of positive integers n, such that the inequalities

$$\frac{9}{17}<\frac{n}{n+k}<\frac{8}{15},$$

hold for *exactly one* positive integer k. Find the largest element of S.

35. The number 2^{29} has exactly 9 distinct digits. Which digit is missing?